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IDENTIFYING INEFFICIENCY IN SMOOTH AGGREGATIVE MODELS OF ECONOMIC GROWTH

A Unifying Criterion*

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The paper establishes a complete characterization of inefficient programs in an aggregative model of economic growth. The main theorem states that a feasible program, satisfying a smoothness condition, is inefficient if and only if (a) the sequence of the value of input is bounded away from zero, and (b) the sequence of the ratio of the share of the primary factor in output to the value of input deteriorates too fast. The unifying nature of this result is established by showing that the well-known characterizations of inefficiency, in the literature, are corollaries of the main theorem.

1. Introduction

In recent years, there have been a number of important contributions toward resolving the following question: what observable characteristic of a growth program signals inefficiency?

The results seem to indicate that the answer depends on the type of technology set which generates these growth programs. Thus, even in the simplest one-good model of growth [where the gross-output function f satisfies $f(0)=0$; f is strictly increasing; f is concave; f is differentiable], there is a diversity of criteria for testing the inefficiency of a program, depending on the conditions that f satisfies, in addition to the basic ones stated above.

I will summarize below the most important of these results.[1] For this

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[1]Many of these results have been generalized, in some form, to many-good cases. McFadden's result has been extended to a simple polyhedral model in Majumdar (1974), and to the case of a convex cone, with output substitution, in Majumdar, Mitra and McFadden (1976). Similarly, the result of Cass has been generalized to many capital goods (and one consumption good) in Cass (1972b), and to many consumption goods in Mitra (1976). The Benveniste–Gale criteria has been generalized to a multi-sectoral case in Benveniste (1976a).

purpose, I shall say that a sequence (α_t) of positive numbers is *bounded away from zero* if $\inf_{t \geqq 0} \alpha_t > 0$. Also, a sequence (β_t) of non-negative numbers will be said to *deteriorate too fast* if $\sum_{t=0}^{\infty} \beta_t < \infty$.

(1) McFadden (1967) has shown that when f is linear in x (i.e., $f(x) = dx$; $d > 0$), inefficiency of a feasible program can be identified with the condition that the sequence of the value of input[2] is bounded away from zero. (2) Cass (1972a) shows that when f is twice continuously differentiable, strictly concave (with $f'' < 0$) and satisfies the end-point conditions: $0 \leq f'(\infty) < 1 < f'(\underline{x}) < \infty$ for some $\underline{x} > 0$, then inefficiency of feasible programs, whose input levels are bounded away from zero, can be identified with the condition that the sequence of the reciprocals of the competitive prices,[3] associated with the program, deteriorates too fast. (3) Benveniste–Gale (1975) have shown that when f is twice differentiable, and there are positive numbers n, N, q, Q, such that $n \leq f'(x), x/f(x) \leq N$, and $q \leq (-f''(x))x^2/f(x) \leq Q$ for $x \geq 0$, then inefficiency of feasible programs is equivalent to the condition that the sequence of the reciprocals of the value of input deteriorates too fast.[4] (4) Benveniste (1976b) has shown that when $\inf_{x \geqq 0} f'(x) > 1$, (i.e., f is 'strongly productive'), then inefficiency of feasible programs is equivalent to the condition that the sequence of the value of input is bounded away from zero.

The main purpose of this paper is to present a *unifying criterion* of inefficiency, which is applicable to all the diverse frameworks mentioned above, and more. The criterion states that a feasible program, satisfying a 'smoothness condition' (see Condition S* in section 3) is inefficient if and only if (a) the sequence of the value of input is bounded away from zero, *and* (b) the sequence of the ratio of the share of the primary factor in output[5] to the value of input deteriorates too fast. (See Theorem 2 for a precise statement.[6])

The usefulness of this result is demonstrated in section 4 by establishing each of the above-mentioned results as corollaries of the unifying criterion.

[2]See section 2 for a definition of this concept.

[3]See section 2 for a definition of this concept.

[4]Actually, Benveniste–Gale consider the production functions to be variable over time, and place these ('elasticity') conditions for $0 \leq x \leq x_t$, where (x, y, c) is the feasible program whose inefficiency (or efficiency) is in question. It is clear, from the following sections, that such aspects can easily be included in the present analysis. However, I have chosen to convey their contribution in the present form, in the interest of notational simplicity, and a unified presentation.

[5]See section 3 for a definition of this concept.

[6]The idea of combining a positive lower bound on input value, like (a), with an infinite series criterion, like (b), was introduced by Benveniste (1976b), where the infinite series criterion was that of Cass (1972a) viz, the reciprocals of the prices deteriorate too fast. However, the theorem of Benveniste only establishes sufficient conditions for a feasible program to be inefficient, and these conditions are not the appropriate necessity conditions for inefficiency, as is clear from section 6.

The points that are exploited here are that (i) the 'smoothness condition' S* can be verified to hold for the appropriate programs in each of these different frameworks, and that (ii) in the models where inefficiency is characterized by Condition (a) [as in McFadden (1967) and Benveniste (1976b)], (a) *implies* (b); and in the models where inefficiency is characterized (essentially) by Condition (b) [as in Cass (1972a) and Benveniste–Gale (1975)], (b) *implies* (a).

The generality of the result is demonstrated in section 6, by showing that the unifying criterion characterizes inefficiency in a 'weakly productive case' [where $f'(x)>1$ for $x\geq 0$ and $\inf_{x\geq 0} f'(x)=1$], which is *not* covered by any of the above mentioned frameworks or criteria. It should be mentioned that the 'weakly productive case' is more interesting, than the 'strongly productive case', considered by Benveniste (1976b). This is because the gross-output function, f, is the sum of depreciated input, and a 'net-output function', and strong productivity implies (even with no depreciation) that the *net* marginal productivity of input remains bounded away from zero, as input levels become infinitely large. Weak productivity allows for the possibility (in the case of no depreciation) that the net marginal productivity goes to zero, as input levels go to infinity.

2. The model

Consider a one-good economy with a technology given by a function, f, from R^+ to itself. The production possibilities consist of inputs x, and outputs $y=f(x)$, for $x\geq 0$.

The following assumptions on f are maintained throughout:

(A.1) $f(0)=0$.
(A.2) f is strictly increasing for $x\geq 0$.
(A.3) f is concave for $x\geq 0$.
(A.4) f is differentiable for $x\geq 0$.

The initial input, $\mathbf{x}$, is considered to be historically given, and positive. A *feasible production program* is a sequence $(x,y)=(x_t,y_{t+1})$ satisfying

$$x_0=\mathbf{x},\quad 0\leq x_t\leq y_t \quad \text{for} \quad t\geqq 1,$$
$$f(x_t)=y_{t+1} \quad \text{for} \quad t\geq 0. \tag{1}$$

The *consumption program* $c=(c_t)$ generated by (x,y) is defined by

$$c_t=y_t-x_t(\geq 0) \quad \text{for} \quad t\geq 1. \tag{2}$$

(x,y,c) is called a *feasible program*, it being understood that (x,y) is a production program, and c is the corresponding consumption program.

A feasible program (x', y', c') *dominates* a feasible program (x, y, c) if $c'_t \geq c_t$ for all $t \geq 1$, and $c'_t > c_t$ for some t. A feasible program (x, y, c) is *inefficient* if there is a feasible program (x', y', c') which dominates it. A feasible program is called *efficient* if it is not inefficient.

The *competitive price* sequence[7] $p = (p_t)$ associated with a feasible program (x, y, c) is given by

$$p_0 = 1, \quad p_{t+1} = p_t / f'(x_t) \quad \text{for} \quad t \geq 0. \tag{3}$$

These are precisely the prices at which *intertemporal profits* are maximized at (x, y, c),

$$w_t = p_{t+1} f(x_t) - p_t x_t \geq p_{t+1} f(x) - p_t x, \qquad x \geq 0, \quad t \geq 0. \tag{4}$$

The *value of input* sequence $v = (v_t)$ associated with a feasible program (x, y, c) is given by

$$v_t = p_t x_t \quad \text{for} \quad t \geq 0. \tag{5}$$

A feasible program (x, y, c) is called *interior* if $\inf_{t \geq 0} x_t > 0$.

A result which will often be used below is that if a feasible program (x, y, c) is inefficient, then $x_t > 0$ for $t \geq 0$, and, hence, $p_t > 0$ for $t \geq 0$. This result is evident from (A.2)–(A.4).

3. Characterization of inefficiency

3.1. A generalization of the Cass technique

A useful starting point in obtaining a generalization of the Cass technique is the following characterization of inefficiency, due to Cass (1972, pp. 203–204).

Lemma 1 (Cass). *Under* (A.1), (A.2), *a feasible program* (x, y, c) *is inefficient iff there is a sequence* (ε_t), *and* $\infty > t_1 \geq 1$, *such that*

$$0 < \varepsilon_t < x_t \quad \textit{for} \quad t \geq t_1, \tag{6}$$

$$\varepsilon_{t+1} \geqq f(x_t) - f(x_t - \varepsilon_t) \quad \textit{for} \quad t \geq t_1. \tag{7}$$

The following partial characterization can be obtained as a corollary:

[7] I shall follow the convention that if $x_{t_1} = 0$ for some finite t_1, and $f'(0) = \infty$, then $p_t = 0$ for $t > t_1$.

Corollary 1. Under (A.1)–(A.4), *if a feasible program* (x, y, c) *is inefficient, then*

$$\inf_{t \geqq 0} p_t x_t > 0. \tag{8}$$

Proof. If (x, y, c) is inefficient, then $x_t > 0$, and, so, $p_t > 0$ for $t \geqq 0$. By Lemma 1, it satisfies (6) and (7). Use (A.3), (A.4) to obtain, from (7),

$$\varepsilon_{t+1} \geqq f'(x_t)\varepsilon_t \quad \text{for} \quad t \geqq t_1. \tag{9}$$

Multiply (9) by $p_{t+1} > 0$, and use (3) to obtain $p_{t+1}\varepsilon_{t+1} \geqq p_t\varepsilon_t$ for $t \geqq t_1$. Thus $p_t\varepsilon_t \geqq p_{t_1}\varepsilon_{t_1}$ for $t \geqq t_1$, and, by (6), $p_t x_t \geqq p_{t_1}\varepsilon_{t_1} > 0$ for $t \geqq t_1$. Hence $p_t x_t \geqq \min[v_0, v_1, \ldots, v_{t_1 - 1}, p_{t_1}\varepsilon_{t_1}] > 0$ for $t \geqq 0$. ∎

Lemma 2. Under (A.1)–(A.4), *if a feasible program* (x, y, c) *satisfies*

$$p_t x_t > 0 \quad \textit{for} \quad t \geqq 0, \tag{10}$$

and

$$\sum_{t=1}^{\infty} (p_t c_t / p_t y_t) < \infty, \tag{11}$$

then (x, y, c) *is inefficient.*

Proof. For $t \geqq 0$, define $z_t = \sum_{s=t}^{\infty} (p_{s+1}c_{s+1}/p_{s+1}y_{s+1})$. Notice, then, that if $z_t = 0$ for some t, then (x, y, c) is inefficient by (10). If not, then $z_t > 0$ for $t \geqq 0$. By (11), we can choose $T \geqq 2$, such that $z_t \leqq \frac{1}{2}$ for $t \geqq T$. Now, define a sequence (x', y', c') in the following way: $x'_t = x_t$ for $0 \leqq t \leqq T-1$, $x'_t = z_t x_t$ for $t \geqq T$, $y'_{t+1} = y_{t+1}$ for $0 \leqq t \leqq T-1$, $y'_{t+1} = f(x'_t)$ for $t \geqq T$, $c'_{t+1} = c_{t+1}$ for $0 \leqq t \leqq T-2$, $c'_{t+1} = y'_{t+1} - x'_{t+1}$ for $t \geqq T-1$. Clearly, $x'_t \geqq 0$, $y'_{t+1} \geqq 0$ for $t \geqq 0$. So, to check that (x', y', c') is feasible, we must show that $c'_{t+1} \geqq 0$ for $t \geqq 0$. By definition, $c'_{t+1} = c_{t+1} \geqq 0$ for $0 \leqq t \leqq T-2$. For $t = T-1$, $c'_{t+1} = y'_{t+1} - x'_{t+1} = y_{t+1} - z_{t+1}x_{t+1} > y_{t+1} - x_{t+1} = c_{t+1}$, using (10). And, for $t \geqq T$,

$$\begin{aligned} c'_{t+1} &= y'_{t+1} - x'_{t+1} = f(x'_t) - x'_{t+1} = f(z_t x_t) - z_{t+1}x_{t+1} \\ &= \{f(z_t x_t)/(z_t x_t)\}(z_t x_t) - z_{t+1}x_{t+1} \geqq [f(x_t)/x_t] z_t x_t - z_{t+1}x_{t+1} \\ &= z_t f(x_t) - z_{t+1}x_{t+1} \geqq (z_t - z_{t+1})y_{t+1} \\ &= \{(p_{t+1}c_{t+1})/(p_{t+1}y_{t+1})\} y_{t+1} = c_{t+1}. \end{aligned}$$

Hence, (x', y', c') is feasible. Also, notice that it dominates (x, y, c), which proves that (x, y, c) is inefficient. ∎

Corollary 2. (A.1)–(A.4), *if a feasible program* (x, y, c) *satisfies*

$$\infty > \sum_{t=1}^{\infty} p_t c_t \geqq \sum_{t=1}^{\infty} p_t c_t', \tag{12}$$

for every feasible program (x', y', c'), *then it is efficient and violates* (8).

Proof Suppose (x, y, c) satisfies (12), but is inefficient. Then $x_t > 0$ for $t \geqq 0$, and $p_t > 0$ for $t \geqq 0$. Furthermore, there is a feasible program (x', y', c') which dominates it. This violates (12), and establishes that (x, y, c) is efficient. Suppose, next, that (x, y, c) satisfies (8). Then, by (12), (x, y, c) satisfies (10) and (11). Hence, by Lemma 2, it is inefficient, a contradiction. So, (x, y, c) violates (8). ∎

Corollary 3. *Under* (A.1)–(A.4), *if an efficient program* (x, y, c) *satisfies*

$$\sum_{t=1}^{\infty} p_t c_t < \infty, \tag{13}$$

then it satisfies (12), *for every feasible program* (x', y', c'), *and violates* (8).

Proof If (x, y, c) satisfies (13), then, by the corollary to the theorem of Cass–Yaari (1971, p. 338), (12) is satisfied. By Corollary 2, (x, y, c) violates (8). ∎

Notice that none of these results are complete characterizations of inefficiency, but they have the advantage that they require only the basic assumptions (A.1)–(A.4). We finally note another corollary, which turns out to be helpful in obtaining the main results (Theorems 1 and 2) of this section.

Corollary 4. *Under* (A.1)–(A.4), *a feasible program* (x, y, c) *is inefficient if and only if for every* λ, *such that* $0 < \lambda \leq 1$, *there is a sequence* (ε_t^λ), *and* $\infty > t_1 \geq 1$, *such that*

$$0 < \varepsilon_t^\lambda < \lambda x_t \quad \textit{for} \quad t \geq t_1 \tag{14}$$

and

$$\varepsilon_{t+1}^\lambda = f(x_t) - f(x_t - \varepsilon_t^\lambda) \quad \textit{for} \quad t \geq t_1. \tag{15}$$

Proof (*Necessity*). If (x, y, c) is inefficient, then there is (ε_t) satisfying (6) and (7). Let a λ be given, satisfying $0 < \lambda \leq 1$. For any $t \geqq t_1$, if there is a number $\hat{a}$ satisfying $0 < \hat{a} \leqq \lambda \varepsilon_t < \lambda x_t$, then there is a number $\hat{b}$, which can be well-defined by $\hat{b}=f(x_t)-f(x_t-\hat{a})$, and which satisfies $0<\hat{b}\leqq\lambda\varepsilon_{t+1}<\lambda x_{t+1}$. To check this, note that $\hat{a}<\lambda x_t$ implies that $\hat{b}$ is well-defined, and $\hat{b}>0$ by (A.2). Also, by (A.3)

$$\begin{aligned} f(x_t)-f(x_t-\hat{a}) &\leqq f(x_t)-f(x_t-\lambda\varepsilon_t) \\ &=\lambda\varepsilon_t\{[f(x_t)-f(x_t-\lambda\varepsilon_t)]/\lambda\varepsilon_t\} \\ &\leqq\lambda\varepsilon_t\{[f(x_t)-f(x_t-\varepsilon_t)]/\varepsilon_t\} \\ &=\lambda[f(x_t)-f(x_t-\varepsilon_t)]=\lambda\varepsilon_{t+1}. \end{aligned}$$

So, $0<\hat{b}\leqq\lambda\varepsilon_{t+1}<\lambda x_{t+1}$.

Now, let $\varepsilon^\lambda_{t_1}=\lambda\varepsilon_{t_1}$, and $\varepsilon^\lambda_{t+1}=f(x_t)-f(x_t-\varepsilon^\lambda_t)$ for $t\geqq t_1$. Then (ε^λ_t) is a well-defined sequence, by the above result, and, furthermore, $0<\varepsilon^\lambda_t\leq\lambda\varepsilon_t<\lambda x_t$, for $t\geqq t_1$.

(*Sufficiency*). If (14) and (15) are satisfied for $\lambda=1$, then (6) and (7) are satisfied for the sequence (ε^1_t). Hence, by Lemma 1, (x,y,c) is inefficient. ∎

Corollary 4 is useful, since it enables one to restrict attention to 'small input decrements', (ε^λ_t), in searching for necessary and sufficient conditions that (6), (7) have a solution.

The general procedure for characterizing feasible programs which admit a solution to (6), (7) is to put sufficient additional structure on f so that $\{(f(x)-f(x-\varepsilon))/\varepsilon\}$ for $0<\varepsilon<\lambda x$, (where $0<\lambda\leq 1$), can be approximated in terms of a separable function of x and ε. To this end, rearrange (15) in the following way for $t\geq t_1$ (given any λ, such that $0<\lambda\leq 1$),

$$\varepsilon^\lambda_{t+1}=f(x_t)-f(x_t-\varepsilon^\lambda_t)=f'(x_t)\varepsilon^\lambda_t\left[1+\left\{\frac{f(x_t)-f(x_t-\varepsilon^\lambda_t)}{\varepsilon^\lambda_t f'(x_t)}-1\right\}\right], \tag{16}$$

and consider the following 'smoothness condition' on the feasible program (x,y,c).

Condition S. For some $0<m\leq M<\infty$, $N<\infty$, *and* $0<\lambda\leq 1$, *there exists a function* $\mu(x)$ *for* $x\geq 0$, *such that*

(a) $0\leq\mu(x_t)\leq N$ *for* $x_t>0$, $t\geq 0$,

and

(b) $m\varepsilon\mu(x_t)/x_t\leq\dfrac{f(x_t)-f(x_t-\varepsilon)}{\varepsilon f'(x_t)}-1\leq M\varepsilon\mu(x_t)/x_t$ *for* $0<\varepsilon<\lambda x_t$, $t\geq 0$.

We can now state:

Theorem 1. Under (A.1)–(A.4), *a feasible program* (x, y, c) *satisfying Condition S, is inefficient if and only if*

$$\inf_{t \geq 0} p_t x_t > 0, \tag{8}$$

and

$$\sum_{s=0}^{\infty} [\mu(x_s)/p_s x_s] < \infty. \tag{17}$$

Proof (*Necessity*). By Corollary 1, (8) is satisfied. To show that (17) is satisfied, we proceed as follows. Let $0 < \lambda \leq 1$ be given by Condition S. By Corollary 4, there is (ε_t^λ) satisfying (14) and (15). Write (15) in the form (16), and perform the following operations on (16) for $t \geqq t_1$ (noting that $x_t > 0$, and $p_t > 0$, for $t \geq 0$). Take reciprocals, and multiply through by $(1/p_{t+1})$ to get

$$(1/p_{t+1}\varepsilon_{t+1}^\lambda) = (1/p_t\varepsilon_t^\lambda)\left\{1 \Big/ \left[1 + \left\{\frac{f(x_t) - f(x_t - \varepsilon_t^\lambda)}{\varepsilon_t^\lambda f'(x_t)} - 1\right\}\right]\right\}.$$

Use the left-hand inequality in Condition S(b) to get

$$(1/p_{t+1}\varepsilon_{t+1}^\lambda) \leq (1/p_t\varepsilon_t^\lambda)(1/[1 + \{m\varepsilon_t^\lambda \mu(x_t)/x_t\}]).$$

Rewrite the right-hand factor as a difference

$$(1/p_{t+1}\varepsilon_{t+1}^\lambda) \leq (1/p_t\varepsilon_t^\lambda)\left[(1/\varepsilon_t^\lambda) - \frac{\{m\mu(x_t)/x_t\}}{1 + \{m\varepsilon_t^\lambda \mu(x_t)/x_t\}}\right]\varepsilon_t^\lambda.$$

Use Condition S(a) and $\varepsilon_t^\lambda < x_t$ to obtain

$$(1/p_{t+1}\varepsilon_{t+1}^\lambda) \leq (1/p_t)[(1/\varepsilon_t^\lambda) - \{m\mu(x_t)/x_t(1 + mN)\}].$$

Simplify the above expression

$$(1/p_{t+1}\varepsilon_{t+1}^\lambda) \leq (1/p_t\varepsilon_t^\lambda) - [m/(1 + mN)](\mu(x_t)/p_t x_t).$$

Sum both sides of the inequality and cancel common terms[8] to get

$$(1/p_{t+1}\varepsilon_{t+1}^\lambda) \leq (1/p_{t_1}\varepsilon_{t_1}^\lambda) - [m/(1 + mN)] \sum_{s=t_1}^{t} [\mu(x_s)/p_s x_s]. \tag{18}$$

[8]This refinement of the original Cass argument (1972a) is due to Benveniste and Gale (1975).

Now, note that for (18) to be consistent with $\varepsilon_t^\lambda > 0$, for $t \geq t_1$ requires that (17) is satisfied.

(*Sufficiency*). Since (8) is satisfied, $x_t > 0$, and $p_t > 0$ for $t \geq 0$. Let $\inf_{t \geq 0} p_t x_t = \delta > 0$, and $\sum_{s=0}^{\infty} [\mu(x_s)/p_s x_s] = Z < \infty$. Let $0 < \lambda \leq 1$ be given by Condition S. Now, define a sequence (δ_t) in the following way. Let

$$\delta_0 = \min(\tfrac{1}{2}/p_0 ZM, \lambda\delta/4p_0),$$

and

$$(1/p_{t+1}\delta_{t+1}) = (1/p_0\delta_0) - M \sum_{s=0}^{t} [\mu(x_s)/p_s x_s] \quad \text{for} \quad t \geqq 0. \tag{19}$$

Then,

$$0 < \delta_t < \lambda x_t \quad \text{for} \quad t \geq 0. \tag{20}$$

To check the left-hand inequality in (20), note that $\delta_0 > 0$. Also, from (19), we have $(1/p_{t+1}\delta_{t+1}) \geqq (1/2p_0\delta_0)$, so that $p_{t+1}\delta_{t+1} > 0$ for $t \geq 0$. Hence $\delta_{t+1} > 0$ for $t \geqq 0$.

To verify the right hand inequality in (20), note that $(1/p_{t+1}\delta_{t+1}) \geqq (1/p_0\delta_0) - MZ = (1/\delta_0) - MZ = (1/2\delta_0) + (1/2\delta_0) - MZ$. Now, since $\delta_0 \leqq 1/2p_0 ZM = 1/2ZM$, so $(1/2\delta_0) \geqq MZ$. Hence, we have $(1/p_{t+1}\delta_{t+1}) \geqq (1/2\delta_0)$, i.e., $p_{t+1}\delta_{t+1} \leqq 2\lambda\delta/4p_0 = \lambda\delta/2 < \lambda\delta \leqq \lambda p_{t+1} x_{t+1}$, by definition of δ and δ_0. Since $p_t > 0$ for $t \geqq 0$, so $\delta_{t+1} < \lambda x_{t+1}$ for $t \geqq 0$. Also, clearly, $\delta_0 \leqq \lambda\delta/4p_0$ so that $p_0\delta_0 \leqq \lambda\delta/4 < \lambda\delta \leqq \lambda p_0 x_0$. Hence, $\delta_0 < \lambda x_0$.

Now from (19), we get

$$\begin{aligned}(1/p_{t+1}\delta_{t+1}) &= (1/p_t\delta_t) - M[\mu(x_t)/p_t x_t] \\ &= (1/p_t)[(1/\delta_t) - \{M\mu(x_t)/x_t\}] \\ &\leq (1/\delta_t) - \frac{M\mu(x_t)/x_t}{1 + \{M\mu(x_t)\delta_t/x_t\}} \\ &= (1/p_t)(1/[1 + \{M\delta_t\mu(x_t)/x_t\}]\delta_t).\end{aligned}$$

Use the right-hand inequality in Condition S(b) to obtain

$$(1/p_{t+1}\delta_{t+1}) \leq (1/p_t\delta_t)\left[1 \Big/ \left\{1 + \left[\frac{f(x_t) - f(x_t - \delta_t)}{\delta_t f'(x_t)} - 1\right]\right\}\right].$$

Multiply through by p_{t+1}, and take reciprocals

$$\delta_{t+1} \geq f'(x_t)\delta_t\left[1+\left\{\frac{f(x_t)-f(x_t-\delta_t)}{\delta_t f'(x_t)}-1\right\}\right],$$

i.e.,

$$\delta_{t+1} \geq f(x_t)-f(x_t-\delta_t) \quad \text{for} \quad t \geq 0. \tag{21}$$

Now, define a sequence (ε_t) as follows:

$$\varepsilon_0 = \delta_0 \quad \text{and} \quad \varepsilon_{t+1} = f(x_t)-f(x_t-\varepsilon_t) \quad \text{for} \quad t \geq 0.$$

Then, by (20) and (21), $0<\varepsilon_t \leq \delta_t < \lambda x_t \leq x_t$ for $t \geq 0$.

Thus (ε_t) satisfies (6) and (7) for $t \geqq 0$, so (x, y, c) is inefficient. ∎

3.2. On the choice of the approximating function

The interesting question, in view of Theorem 1, is what to use for the approximating function, $\mu(x)$. A reasonable restriction on μ would be that it is 'observable'; that is, it could be calculated along a feasible program without knowing the function f itself. This means that it would have to be some elementary function of $x_t, y_{t+1}, r_t = f'(x_t), p_t, u_t = f(x_t) - x_t f'(x_t)$ etc., since it is obvious that these are observable magnitudes.

Cass (1972a) used $\mu(x) = x/f'(x)$, while Benveniste–Gale (1975) used $\mu(x) = 1$. Both choices have 'end-point' problems. For example, the choice of Cass fails for $f'(0) = \infty$, and the choice of Benveniste–Gale fails for $f'(0) < \infty$. Both choices fail for $f'(\infty) \geq 1$. My own choice in an earlier work was $\mu(x) = -f''(x)x/f'(x)$ [see Mitra (1975)], which has the advantage of covering cases where $f'(\infty) \geq 1$, but the disadvantage of not being observable. (Besides, twice continuous differentiability becomes indispensable.) This choice is however, closely related to the share of the primary factor in output, which is observable, and which, for reasons that will be evident in the next section, unifies all the well-known results.[9] To see the relationship between the two choices, note that if f is twice continuously differentiable, then

$$\left\{\frac{f(x)-f(x-\varepsilon)}{\varepsilon f'(x)}-1\right\} = \frac{[-f''(x)]\varepsilon}{2f'(x)} + \frac{R}{\varepsilon f'(x)},$$

where R is the second-order remainder. This suggests $\{[-f''(x)]x/f'(x)\}$ as a 'natural' choice of $\mu(x)$. But, essentially the same approximation is effected

[9]This was suggested to me by David Cass, in response to my earlier work.

by using $W(x)$ [defined in (22) below] as the choice of $\mu(x)$. To see this, note that $f(0)-f(x)=(-x)f'(x)+\frac{1}{2}f''(x)x^2+\bar{R}$, where $\bar{R}$ is the second-order remainder. Hence, using (A.1),

$$f(x)-xf'(x)=\tfrac{1}{2}[-f''(x)]x^2-\bar{R},$$

so

$$\left[1-\frac{xf'(x)}{f(x)}\right]=\frac{1}{2}\frac{[-f''(x)]}{f(x)}x^2-\frac{\bar{R}}{f(x)}$$

$$=\frac{1}{2}\frac{[-f''(x)]x}{f'(x)}\left[\frac{f'(x)x}{f(x)}\right]-\frac{\bar{R}}{f(x)}.$$

Since $f'(x)\cdot x/f(x)\leqq 1$ [see P(ii) in section 4], for production functions satisfying $\inf_{x\geqq 0}f'(x)x/f(x)>0$ [see (BG.2) in section 4, in this connection], $W(x)$ is 'essentially' the same approximating function as $[-f''(x)]x/f'(x)$. Also, note that the limit of both choices of $\mu(x)$, as $x\to 0$, is the same, by l'Hopital's rule.

Define the share of the primary factor in output as

$$W(x)=1-[f'(x)x/f(x)] \quad \text{for} \quad x>0,$$

and

$$W(x)=0 \quad \text{for} \quad x=0. \tag{22}$$

Let us choose the approximating function $\mu(x)=W(x)$ for $x\geq 0$, and restate the 'smoothness condition' on a feasible program (x,y,c) as:

Condition S. For some* $0<m\leq M<\infty$, *and* $0<\lambda\leq 1$,

$$m\varepsilon W(x_t)/x_t\leq\{[f(x_t)-f(x_t-\varepsilon)]/\varepsilon f'(x_t)\}-1$$
$$\leq M\varepsilon W(x_t)/x_t \quad \textit{for} \quad 0<\varepsilon<\lambda x_t, \quad t\geq 0.$$

We can prove, using Theorem 1, the unifying criterion of inefficiency.

Theorem 2. Under (A.1)–(A.4), *a feasible program* (x,y,c) *satisfying Condition S* is inefficient if and only if*

$$\inf_{t\geqq 0} p_t x_t>0, \tag{8}$$

and

$$\sum_{s=0}^{\infty} [W(x_s)/p_s x_s] < \infty. \tag{23}$$

Remarks. (1) Condition S* is satisfied by feasible programs generated by the class of gross-output functions: $f(x)=ax^{\alpha}+bx$, where $a,b\geq 0$, $(a+b)>0$, and $0<\alpha<1$. The importance of this parametric form should be obvious as it permits both $f'(\infty)<1$, and $f'(\infty)\geq 1$. Condition S is not satisfied, for Cass's or Benveniste–Gale's choices of μ, by this class of functions.

To check the validity of the remark, we note, first, that with $b\geq 1$, $f'(\infty)=b\geq 1$; and with $b<1, f'(\infty)=b<1$, since $f'(x)=\alpha a x^{\alpha-1}+b$, and $\alpha a x^{\alpha-1}\to 0$ as $x\to\infty$.

Condition S* is satisfied for $\lambda=\frac{1}{2}, m=\alpha/2$, and $M=(\frac{1}{2})^{\alpha-1}$. To check this, note that for $x>0$,

$$\frac{[f(x)-f(x-\varepsilon)]}{\varepsilon f'(x)}-1=\frac{\{\varepsilon f'(x)+\frac{1}{2}[-f''(\S)]\varepsilon^2\}}{\varepsilon f'(x)}-1$$
$$=\frac{-[f''(\S)]\varepsilon}{2f'(x)}=\frac{[-f''(\S)]x}{2f'(x)}\left(\frac{\varepsilon}{x}\right).$$

Also, for $f(x)=ax^{\alpha}+bx$, $f'(x)x/f(x)=(\alpha a x^{\alpha}+bx)/(ax^{\alpha}+bx)$, and $W(x)=(1-\alpha)ax^{\alpha}/(ax^{\alpha}+bx)$. For $\lambda=\frac{1}{2}$, note that $\varepsilon<\lambda x=\frac{1}{2}x$ implies $(x-\varepsilon)>\frac{1}{2}x$, i.e., $x\geqq\S>\frac{1}{2}x$. Hence, $[-f''(\S)]=\alpha(1-\alpha)a\S^{\alpha-2}\geqq\alpha(1-\alpha)ax^{\alpha-2}$, and $[-f''(\S)]\leqq\alpha(1-\alpha)a(\frac{1}{2})^{\alpha-2}x^{\alpha-2}$.

To check the left-hand inequality of Condition S*, note that

$$\frac{[-f''(\S)]x}{2f'(x)}\geqq\frac{\alpha(1-\alpha)ax^{\alpha-1}}{2[\alpha a x^{\alpha-1}+b]}\geqq\frac{\alpha(1-\alpha)ax^{\alpha-1}}{2[ax^{\alpha-1}+b]}=\left(\frac{\alpha}{2}\right)\frac{(1-\alpha)ax^{\alpha}}{[ax^{\alpha}+bx]}$$
$$=\frac{\alpha}{2}W(x)=mW(x).$$

Hence

$$\frac{[f(x)-f(x-\varepsilon)]}{\varepsilon f'(x)}-1=\frac{[-f''(\S)]}{2f'(x)}\left(\frac{\varepsilon}{x}\right)\geqq m\,W(x)\varepsilon/x,$$

for $0<\varepsilon<\lambda x$.

To check the right-hand inequality of Condition S*, note that

$$\frac{[-f''(\S)]}{2f'(x)}\leqq\frac{\alpha(1-\alpha)a(\frac{1}{2})^{\alpha-2}x^{\alpha-1}}{2[\alpha a x^{\alpha-1}+b]}\leqq\frac{(\frac{1}{2})^{\alpha-2}\alpha(1-\alpha)ax^{\alpha-1}}{2\alpha[ax^{\alpha-1}+b]}$$

$$\leqq \frac{(\frac{1}{2})^{\alpha-2}(1-\alpha)ax^{\alpha}}{2[ax^{\alpha}+bx]} = \left(\frac{1}{2}\right)^{\alpha-1} W(x) = M\,W(x).$$

Hence,

$$\frac{[f(x)-f(x-\varepsilon)]}{\varepsilon f'(x)} - 1 = \frac{[-f''(\S)]x}{2f'(x)}\left(\frac{\varepsilon}{x}\right) \leq M\,W(x)\varepsilon/x,$$

for $0<\varepsilon<\lambda x$.

To check that the choices of $\mu(x)$ of Cass or Benveniste–Gale do not work in general for this class of functions, let $a=b=1$. Then

$$\frac{f(x)-f(x-\varepsilon)}{\varepsilon f'(x)} - 1 = \frac{-f''(\S)\varepsilon}{2f'(x)} = \left(\frac{1}{2}\right)\left(\frac{\varepsilon}{x}\right)\frac{[-f''(\S)]x}{f'(x)}.$$

Let $0<\lambda<1$ and $0<\varepsilon<\lambda x$. Then

$$\frac{[-f''(\S)]x}{f'(x)} \leqq \frac{a(1-\alpha)(1-\lambda)^{\alpha-2}x^{\alpha-1}}{\alpha x^{\alpha-1}+1},$$

which converges to zero as $x\to\infty$, so that with $\mu(x)=1$, Condition S* is violated. Also $[-f''(\S)] \leqq \alpha(1-\alpha)(1-\lambda)^{\alpha-2}x^{\alpha-2}$ which converges to zero as $x\to\infty$, so that with $\mu(x)=x/f'(x)$, Condition S* is violated.

(2) Note that only the left-hand inequality of Condition S* is required for the necessity part of Theorem 2, and only the right-hand inequality for the sufficiency part. This is obvious from the proof of Theorem 1.

An alternative statement of Theorem 2 can be obtained fairly easily. It is useful, particularly for comparison with the partial characterization result of Lemma 2.

Corollary 5. Under (A.1)–(A.4), *a feasible program* (x, y, c), *satisfying Condition S*, is inefficient if and only if*

$$\inf_{t\geqq 0} p_t x_t > 0, \tag{8}$$

and

$$\sum_{t=0}^{\infty} \frac{(p_{t+1}c_{t+1})}{(p_{t+1}y_{t+1})(p_{t+1}x_{t+1})} < \infty. \tag{24}$$

Proof (*Necessity*). If (x, y, c) satisfies Condition S*, and is inefficient, then by Theorem 2, (8) and (23) are satisfied. Note that for $t\geqq 0$, we have

$$p_{t+1}c_{t+1}=p_{t+1}y_{t+1}-p_{t+1}x_{t+1}.$$

Dividing through by $(p_{t+1}y_{t+1})(p_{t+1}x_{t+1})$,

$$\begin{aligned}&\{(p_{t+1}c_{t+1})/(p_{t+1}y_{t+1})(p_{t+1}x_{t+1})\}\\&\quad=\{1/(p_{t+1}x_{t+1})\}-\{1/(p_{t+1}y_{t+1})\}\\&\quad=\{1/(p_{t+1}x_{t+1})\}-\{1/(p_tx_t)\}+\{W(x_t)/p_tx_t\}.\end{aligned}$$

Summing from $t=0$ to $t=T$,

$$\begin{aligned}&\sum_{t=0}^{T}\{(p_{t+1}c_{t+1})/(p_{t+1}y_{t+1})(p_{t+1}x_{t+1})\}\\&\quad\leqq\{1/(p_{T+1}x_{T+1})\}+\sum_{t=0}^{T}\{W(x_t)/p_tx_t\}.\end{aligned}$$

Hence, by (8) and (23), (24) is satisfied.

(*Sufficiency*). Suppose (x,y,c) satisfies Condition S*, and (8), (24). By the calculations used in the necessity part, we have for $T\geqq 0$,

$$\begin{aligned}&\sum_{t=0}^{T}\{W(x_t)/p_tx_t\}\\&\quad\leqq\{1/(p_0x_0)\}+\sum_{t=0}^{T}\{(p_{t+1}c_{t+1})/(p_{t+1}y_{t+1})(p_{t+1}x_{t+1})\}.\end{aligned}$$

Hence, using (24), we have (23). By Theorem 2, (x,y,c) is inefficient. ∎

4. Four applications

In this section, I shall apply Theorem 2 to establish the theorems of Cass (1972a), Benveniste–Gale (1975), McFadden (1967), and Benveniste (1976b), in that order.

In establishing the proofs in this section, and the next, use will be made of the following properties of f, which hold, under (A.1)–(A.4):

P(i) *If* $x\geqq\tilde{x}\geqq 0$, *then* $f'(x)\leqq f'(\tilde{x})$.
P(ii) *For* $x>0, f'(x)\leqq f(x)/x$.
P(iii) *If* $x\geqq\tilde{x}>0$, *then* $[f(x)/x)\leqq[f(\tilde{x})/\tilde{x}]]$.

P(i) clearly holds if $x=\tilde{x}$. If $x>\tilde{x}$, use (A.3) and (A.4) to obtain $f'(x)(x-\tilde{x})\leqq f(x)-f(\tilde{x})\leqq f'(\tilde{x})(x-\tilde{x})$, which yields $f'(x)\geqq f'(\tilde{x})$. P(ii) holds since $f(x)$

$=f(x)-f(0)\geqq f'(x)x$ by (A.1), (A.3) and (A.4). P(iii) holds, since by (A.3), $f(\tilde{x})=f[(\tilde{x}/x)x+(1-\tilde{x}/x)0]\geqq(\tilde{x}/x)f(x)+(1-\tilde{x}/x)f(0)=(\tilde{x}/x)f(x)$, by (A.1). Hence, $f(\tilde{x})/\tilde{x}\geq f(x)/x$.

4.1. The Cass criterion

Cass considers a model in which f satisfies, in addition to (A.1), (A.2), the following conditions:

(C.1) $f(x)$ is twice continuously differentiable for $x\geqq 0$.
(C.2) $f(x)$ is strictly concave, with $f''<0$, for $x\geq 0$.
(C.3) $f(x)$ satisfies the end-point conditions: $0\leq f'(\infty)<1<f'(\underline{x})<\infty$ for some $\underline{x}>0$.

Theorem 3 (Cass). Under (A.1), (A.2), (C.1)–(C.3), *an interior program* (x,y,c) *is inefficient if and only if*

$$\sum_{t=0}^{\infty}(1/p_t)<\infty. \tag{25}$$

Proof. Clearly, (C.1), (C.2) ensure that (A.3), (A.4) are satisfied. We shall establish that any interior program (x,y,c) satisfies Condition S*. Since (x,y,c) is interior and (C.3) holds, so there exist $0<k\leq K<\infty$, such that $k\leq x_t\leq K$ for $t\geqq 0$. By (C.1) and (C.2), there are positive numbers $h,H,l,L,\hat{w}$ such that for $\frac{1}{2}k\leq x\leq K$, $h\leq f'(x)\leq H$, $l\leq -f''(x)\leq L$, and $W(x)\geq\hat{w}$. Then, by choosing $\lambda=\frac{1}{2}$, $M=KL/2h\hat{w}$ and $m=kl/2H$, Condition S* is satisfied.

To check this, note that

$$\frac{f(x_t)-f(x_t-\varepsilon)}{\varepsilon f'(x_t)}-1=\frac{1}{2}\frac{[-f''(\S_t)]\varepsilon}{f'(x_t)},$$

where

$$x_t-\varepsilon\leqq\S_t\leqq x_t \text{ (by Taylor's expansion)}=\frac{1}{2}\frac{[-f''(\S_t)]x_t}{f'(x_t)}\left(\frac{\varepsilon}{x_t}\right).$$

Since $\lambda=\frac{1}{2}$, and $\varepsilon<\lambda x_t$ so $\varepsilon<\frac{1}{2}x_t$, so that $(x_t-\varepsilon)>\frac{1}{2}x_t$, and $\S_t\geqq\frac{1}{2}x_t\geqq k/2$. Also, $\S_t\leqq x_t\leqq K$, so $k/2\leqq\S_t\leqq K$.

Hence, using the bounds on the first and second-derivatives,

$$\frac{1}{2}\frac{[-f''(\S_t)]x_t}{f'(x_t)}\geqq\frac{1}{2}\frac{lk}{H}\geqq mW(x_t)\quad[\text{since } W(x)\leqq 1],$$

and

$$\frac{1}{2}\frac{[-f''(\S_t)]x_t}{f'(x_t)} \le \frac{1}{2}\frac{LK}{h} \leqq \frac{1}{2}\frac{LK\hat{w}}{h\hat{w}} \leqq MW(x_t).$$

(*Necessity*). By Theorem 2, (23) is satisfied. Since (x, y, c) is interior, $W(x_t) \geq \hat{w}$, and $(1/x_t) \geq (1/K)$ for $t \geq 0$. Hence, (25) is satisfied.

(*Sufficiency*). Since (x, y, c) is interior, $(1/x_t) \leq (1/k)$, $t \geq 0$. And, clearly, $W(x_t) \leq 1$, for $t \geq 0$. So, (25) implies that (23) holds. Also, (25) clearly implies that (8) holds, since $x_t \geq k > 0$, for $t \geq 0$. Hence, by Theorem 2, (x, y, c) is inefficient. ∎

4.2. *The Benveniste–Gale criterion*

Benveniste–Gale consider a model, in which f satisfies the following conditions:

(BG.1) $f(x)$ is twice differentiable for $x \geq 0$.

(BG.2) There are positive numbers n, N, q, Q such that

$$n \leq [f'(x)x/f(x)] \leq N; \qquad q \leq [-f''(x)x^2/f(x)] \leq Q \quad \text{for} \quad x \geq 0.$$

Theorem 4 (*Benveniste–Gale*). *Under* (BG.1), (BG.2), *a feasible program* (x, y, c) *is inefficient if and only if*

$$\sum_{t=0}^{\infty} (1/p_t x_t) < \infty. \tag{26}$$

Proof. It can be checked that (BG.1), (BG.2) imply that the assumptions (A.1)–(A.4) are satisfied. Also, note that $1 \geq W(x) \geq \frac{1}{2}q$ for $x > 0$. To check the right-hand inequality, note that $f(x) = f(x) - f(0) = f'(x)x + \frac{1}{2}[-f''(\zeta)]x^2$ (by Taylor's expansion), where $0 < \zeta < x$. That is,

$$1 = \frac{f'(x)x}{f(x)} + \frac{1}{2}\frac{[-f''(\zeta)]x^2}{f(x)},$$

or

$$W(x) = \frac{1}{2}\frac{[-f''(\zeta)]x^2}{f(x)} = \frac{1}{2}\frac{[-f''(\zeta)]x}{f(x)/x]} \geqq \frac{1}{2}\frac{[-f''(\zeta)]\zeta}{f(\zeta)/\zeta]} \text{ [by P(iii)]}$$

$$= \frac{1}{2}\frac{[-f''(\zeta)]\zeta^2}{f(\zeta)} \geq \frac{1}{2}q.$$

So Condition S* is satisfied for any feasible program (x, y, c) by choosing $\lambda = \frac{1}{2}$, $m = \frac{1}{2}q$ and $M = 4Q/nq$. To check this, note that

$$\frac{f(x_t)-f(x_t-\varepsilon)}{\varepsilon f'(x_t)}-1=\frac{1}{2}\frac{[-f''(\S_t)]\varepsilon}{f'(x_t)},$$

where

$$x_t-\varepsilon \leqq \S_t \leqq x_t \text{ (by Taylor's expansion)}=\frac{1}{2}\frac{[-f''(\S_t)]x_t}{f'(x_t)}\left(\frac{\varepsilon}{x_t}\right).$$

Since $\lambda=\frac{1}{2}$, and $\varepsilon<\lambda x_t$ so $\varepsilon<\frac{1}{2}x_t$, so that $(x_t-\varepsilon)>\frac{1}{2}x_t$, and $\S_t>\frac{1}{2}x_t$, i.e., $\S_t^2>\frac{1}{4}x_t^2$. Hence, using the bounds in (BG.2),

$$\frac{1}{2}\frac{[-f''(\S_t)]x_t}{f'(x_t)}=\frac{1}{2}\frac{[-f''(\S_t)]x_t^2}{f(x_t)}\left[1\Big/\left\{\frac{f'(x_t)x_t}{f(x_t)}\right\}\right]$$

$$\leqq\frac{1}{2}\frac{[-f''(\S_t)]4\S_t^2}{f(\S_t)}\left(\frac{1}{n}\right)$$

$$\leqq\frac{4Q}{2n}=\left(\frac{4Q}{nq}\right)\frac{q}{2}\leqq\left(\frac{4Q}{nq}\right)W(x_t),$$

and

$$\frac{1}{2}\frac{[-f''(\S_t)]x_t}{f'(x_t)}\geqq\frac{1}{2}\frac{[-f''(\S_t)]}{f(x_t)}x_t^2 \text{ [by P(ii)]}$$

$$\geqq\frac{1}{2}\frac{[-f''(\S_t)]\S_t x_t}{f(\S_t)} \text{ [by P(iii)]}$$

$$\geqq\frac{1}{2}\frac{[-f''(\S_t)]\S_t^2}{f(\S_t)}\geqq\frac{q}{2}\geqq\frac{1}{2}q\, W(x_t).$$

(*Necessity*). By Theorem 2, (23) is satisfied. Since $x_t>0$ for $t\geq 0$, so $W(x_t)\geq\frac{1}{2}q$, so that (26) holds.

(*Sufficiency*). (26) implies that (23) is satisfied, since $W(x_t)\leq 1$, for $t\geq 0$. Also, (26) clearly implies (8). Hence, by Theorem 2, (x,y,c) is inefficient. ▮

4.3. The McFadden result

McFadden considers a simple linear model, where f satisfies

(M.1) $f(x)=dx$ for $x\geq 0$, where $d>0$.

Theorem 5 (McFadden). Under (M.1), *a feasible program* (x, y, c) *is inefficient iff* (8) *is satisfied.*

Proof. Clearly, (M.1) implies that (A.1)–(A.4) are satisfied. Also, since $W(x)=0$ for $x \geq 0$, any feasible program (x, y, c) satisfies Condition S*, for $\lambda = \mathrm{m} = M = 1$.

(*Necessity*). By Theorem 2, (8) holds.

(*Sufficiency*). Since (8) holds, and $W(x_t)=0$ for $t \geq 0$, so (23) holds also. By Theorem 2, (x, y, c) is inefficient. ▌

4.4. The Benveniste result

This is the only result in which Theorem 2 is not *directly* applicable. However, as we shall see, the same technique of proof (as in Theorem 1) can be used to establish this result.

Benveniste considers a 'strongly productive' model in which $f(x)$ satisfies (A.1)–(A.4), and the following condition:

(B.1) $$\inf_{x \geqq 0} f'(x) = \hat{a} > 1.$$

Theorem 6 (Benveniste). Under (A.1)–(A.4), (B.1), *a feasible program* (x, y, c) *is inefficient if and only if it satisfies* (8).

Proof (*Necessity*). Obvious from Corollary 1.

(*Sufficiency*). Consider the 'pure accumulation program' $(\bar{x}, \bar{y}, \bar{c})$ given by $\bar{x}_{t+1} = f(\bar{x}_t)$ for $t \geqq 0$. Then, there is $V < \infty$, such that $\bar{p}_t \bar{x}_t \leq V$, for $t \geq 0$. To check this, note that for $t \geqq 1$, $(\bar{x}_{t+1} - \bar{x}_t) = f(\bar{x}_t) - f(\bar{x}_{t-1}) \leqq f'(\bar{x}_{t-1})(\bar{x}_t - \bar{x}_{t-1})$. Iterating on this relationship, we have, for

$$t \geqq 1, (\bar{x}_{t+1} - \bar{x}_t) \leqq \prod_{s=0}^{t-1} f'(x_s)(\bar{x}_1 - \bar{x}_0),$$

or

$$\bar{p}_{t+1}(\bar{x}_{t+1} - \bar{x}_t) \leqq (\bar{x}_1 - \bar{x}_0)/f'(\bar{x}_t) \leqq (\bar{x}_1 - \bar{x}_0)/\hat{a} = [f(\mathbf{x}) - \mathbf{x}]/\hat{a} = \hat{b}.$$

Hence,

$$\bar{p}_{t+1}\bar{x}_{t+1} \leqq \bar{p}_{t+1}\bar{x}_t + \hat{b} = [(\bar{p}_t \bar{x}_t)/\hat{a}] + \hat{b}.$$

That is,

$$\bar{p}_{t+1}\bar{x}_{t+1} \leqq [(\bar{p}_0 \bar{x}_0)/\hat{a}^{t+1}] + \sum_{s=0}^{t} [\hat{b}/\hat{a}^s] \leqq \mathbf{x} + [(\hat{a}\hat{b})/(\hat{a}-1)] < \infty.^{10}$$

[10] This was first proved by Benveniste (1976b, p. 340). The proof presented here is slightly different and slightly shorter.

Then, since $p_t \leqq \bar{p}_t$ for $t \geqq 0$, so we have, for $T \geq 1$,

$$\sum_{t=1}^{T} p_t c_t \leq \sum_{t=1}^{T} \bar{p}_t c_t = \sum_{t=1}^{T} \bar{p}_t (c_t - \bar{c}_t) \leq \bar{p}_T \bar{x}_T \leq V.$$

Hence

$$\sum_{t=1}^{\infty} p_t c_t = \lim_{T \to \infty} \sum_{t=1}^{T} p_t c_t$$

exists and

$$\sum_{t=1}^{\infty} p_t c_t \leq V.$$ [11]

Also, note that, for $t \geq 0$,

$$\begin{aligned} p_{t+1} c_{t+1} &= p_{t+1}[f(x_t) - x_{t+1}] \\ &= p_{t+1} f'(x_t) x_t + p_{t+1}[f(x_t) - f'(x_t) x_t] - p_{t+1} x_{t+1} \\ &= p_t x_t + w_t - p_{t+1} x_{t+1} \end{aligned}$$

(where w_t denotes $p_{t+1}[f(x_t) - f'(x_t)x_t]$). Hence

$$\sum_{t=0}^{T} p_{t+1} c_{t+1} = p_0 x_0 + \sum_{t=0}^{T} w_t - p_{T+1} x_{T+1},$$

and

$$\sum_{t=0}^{T} w_t \leq \sum_{t=0}^{T} p_{t+1} c_{t+1} + p_{T+1} x_{T+1} \leq 2V,$$

for $T \geq 1$. Thus

$$\sum_{t=0}^{\infty} w_t = \lim_{T \to 0} \sum_{t=0}^{T} w_t$$

exists, and

$$\sum_{t=0}^{\infty} w_t \leq 2V.$$

[11]The proof can be completed very briefly from this step, by appealing to Corollary 3. I have chosen to present a different proof, in the interest of exhibiting the wide applicability of the technique used in section 3.

We can now show that (8) implies that (23) is satisfied. Let

$$\inf_{t \geqq 0} p_t x_t = \delta > 0.$$

Then,

$$W(x_t) = (w_t/p_{t+1} f(x_t)) \leq (w_t/p_t x_t).$$

Hence,

$$\sum_{t=0}^{\infty} (W(x_t)/p_t x_t) \leq \sum_{t=0}^{\infty} [w_t/(p_t x_t)^2] \leq (2V/\delta^2) < \infty.$$

Note that, from above, $\sum_{t=0}^{\infty} W(x_t) < \infty$, so that $\inf_{t \geqq 0}[f'(x_t)x_t/f(x_t)] = \theta > 0$. Let t_1 be chosen such that $\hat{Z} = \sum_{s=t_1}^{t} [W(x_s)/p_s] \leq (\theta/8V)$, and define a sequence (δ_t), where $t \geq t_1$, as follows: $\delta_{t_1} = \frac{1}{4}\delta/p_{t_1}$ and

$$(1/p_{t+1}\delta_{t+1}) = (1/p_{t_1}\delta_{t_1}) - (4V/\theta\delta) \sum_{s=t_1}^{t} [W(x_s)/p_s x_s] \quad \text{for } t \geqq t_1.$$

Then by choosing $\lambda = 1$, and $M = (4V/\theta\delta)$, it can be checked that the *right-hand inequality* of Condition S* is satisfied, if ε is replaced by δ_t, for $t \geq t_1$.[12]

[12]It is simple to check that $0 < \delta_t < x_t$ for $t \geq t_1$. Also since

$$(1/p_{t+1}\delta_{t+1}) \leq (1/p_{t_1}\delta_{t_1}),$$

so

$$p_{t+1}\delta_{t+1} \geq V(p_{t_1}\delta_{t_1}/V) \geq (p_{t_1}\delta_{t_1}/V) p_{t+1} x_{t+1};$$

or,

$$\delta_{t+1} \geq (\delta/4V) x_{t+1}.$$

Hence,

$$(x_t/\delta_t) \leq (4V/\delta) \quad \text{for} \quad t \geq t_1.$$

By concavity of f,

$$[f(x_t) - f(x_t - \delta_t)]/\delta_t \leq f(x_t)/x_t \quad \text{for} \quad t \geq t_1,$$

and hence,

$$\begin{aligned}[\{f(x_t) - f(x_t - \delta_t)\}/\delta_t f'(x_t)] - 1 &\leq [f(x_t)/x_t f'(x_t)] - 1 \\ &= [f(x_t)/x_t f'(x_t)] W(x_t)(\delta_t/x_t)(x_t/\delta_t) \\ &\leq (\delta_t/x_t) W(x_t)(1/\theta)(x_t/\delta_t) \\ &\leq (4V/\theta\delta)(\delta_t/x_t) W(x_t) = M(\delta_t/x_t) W(x_t).\end{aligned}$$

Note then that the sufficiency proof of Theorem 1 holds exactly, and there is a sequence (ε_t), where $t \geq t_1$, satisfying (6) and (7). Hence, (x, y, c) is inefficient. ∎

5. Two cases where the unifying criterion fails

It is only fair to point out cases where characterization of inefficiency in terms of (8) and (23) fails. The necessity part fails when the left-hand inequality of Condition S* is violated, and the sufficiency part fails, when the right-hand inequality of Condition S* is violated. I shall present below, an example of each of these possibilities. The examples, by showing cases where Condition S* fails, also help, to a certain extent, in understanding the implications of the rather complicated-looking smoothness condition.

Example 1.[13] Let

$$f(x) = 2x^{1/2} \quad \text{for} \quad 0 \leqq x \leqq 1,$$

$$f(x) = 1 + x \quad \text{for} \quad 1 \leqq x \leqq 3,$$

$$f(x) = (4/3^{3/4})x^{3/4} \quad \text{for} \quad x \geqq 3.$$

Then $f(x)$ satisfies (A.1)–(A.4). Consider the feasible program (x, y, c) given by $x_t = \boldsymbol{x} = 2, y_{t+1} = 3$ and $c_{t+1} = 1$ for $t \geqq 0$. Then, $p_t = 1$ for $t \geqq 0$, and $p_t x_t = 2$ for $t \geqq 0$. $W(x_t) = 1 - x_t/(1 + x_t) = 1/(1 + x_t) = \frac{1}{3}$ for $t \geqq 0$. Hence $W(x_t)/p_t x_t = \frac{1}{6}$ for $t \geqq 0$, and (23) fails. However, the program (x, y, c) is clearly inefficient, since $(\tilde{x}, \tilde{y}, \tilde{c})$ defined by $\tilde{x}_0 = \boldsymbol{x} = 2$, $\tilde{x}_t = 1$ for $t \geqq 1$; $\tilde{y}_1 = 3$, $\tilde{y}_{t+1} = 2$ for $t \geqq 1$; $\tilde{c}_1 = 2$, $\tilde{c}_{t+1} = 1$ for $t \geqq 1$; is feasible, and dominates (x, y, c). Thus, inefficiency, in this framework, does not imply (23). This is because, here, the left-hand inequality of Condition S* fails. To check this, pick any λ, such that $0 < \lambda \leqq 1$, and let $\varepsilon = \lambda/2$. Note that

$$f(x_t) - f(x_t - \varepsilon) = x_t - (x_t - \varepsilon) = \varepsilon = \lambda/2.$$

Hence,

$$\{[f(x_t) - f(x_t - \varepsilon)]/[\varepsilon f'(x_t)]\} - 1 = [\lambda/2]/[\lambda/2] - 1 = 0 \quad \text{for} \quad t \geqq 0.$$

Also, note that

$$(\varepsilon/x_t)W(x_t) = ([\lambda/2]/2)(1/6) = \lambda/24 \quad \text{for} \quad t \geqq 0.$$

[13] This is essentially the same example as the one presented in Cass (1972a, p. 222), except that I have specified the function f more explicitly.

Hence, there are no numbers λ, m satisfying the left-hand inequality of Condition S*, for the program (x, y, c).

Example 2. Let

$$f(x)=2x^{1/2} \quad \text{for} \quad 0 \leqq x \leqq 1,$$

$$f(x)=(1+x)-(2/3)(x-1)^{3/2} \quad \text{for} \quad 1 \leqq x \leqq 25/16,$$

$$f(x)=(5/8)x^{1/2}+3/2 \quad \text{for} \quad x \geqq 25/16.$$

Then, $f(x)$ satisfies (A.1)–(A.4). Consider the feasible program (x, y, c) given by $x_0 = \boldsymbol{x} = 1$, $x_t = 1+(2t+1)^2/(t+1)^4$ for $t \geqq 1$; $y_{t+1} = f(x_t)$ for $t \geqq 0$, $c_{t+1} = y_{t+1} - x_{t+1}$ for $t \geqq 0$. Then, $f'(x_0)=1$, $f'(x_t)=1-(x_t-1)^{1/2}=1-(2t+1)/(t+1)^2 = t^2/(t+1)^2$ for $t \geqq 1$. Hence, $p_0 = 1$, and $p_t = t^2$ for $t \geqq 1$. Then, clearly (8) is satisfied. And, since $W(x_t) \leqq 1$ for $t \geqq 0$, and $p_t x_t \geqq t^2$ for $t \geqq 1$, so $\sum_{t=0}^{\infty} (W(x_t)/p_t x_t) < \infty$, so (23) is satisfied. However, it can be shown that (x, y, c) is efficient, so that, in this framework, (8), (23) do not imply inefficiency.

To verify the efficiency of (x, y, c), suppose, on the contrary, that (x, y, c) is inefficient. Then, there is a sequence (ε_t) and $1 \leqq t_1 < \infty$, satisfying (6) and (7). Define $x'_t = x_t$ for $0 \leqq t < t_1$. $x'_t = x_t - \varepsilon_t$ for $t \geqq t_1$; $y'_{t+1} = f(x'_t)$, $c'_{t+1} = y'_{t+1} - x'_{t+1}$ for $t \geqq 0$. Then, (x', y', c') is feasible, and $c'_t = c_t$ for $1 \leqq t < t_1$, $c'_{t_1} = c_{t_1} + \varepsilon_{t_1} > c_{t_1}$, and for

$$t > t_1, c'_t = f(x'_{t-1}) - x'_t = f(x_{t-1} - \varepsilon_{t-1}) - (x_t - \varepsilon_t)$$
$$= f(x_{t-1} - \varepsilon_{t-1}) - f(x_{t-1}) + f(x_{t-1}) - x_t + \varepsilon_t$$
$$= -\varepsilon_t + c_t + \varepsilon_t = c_t.$$

It can be checked that for $t \geqq 3$, $c_{t+1} > 1$, so that $c'_{t+1} > 1$ for $t \geqq 3$. Hence, for $t \geqq 3$, $x'_t > 1$. Otherwise, if $x'_t \leqq 1$, for some $t \geqq 3$, then it is impossible to maintain the consumption level of (x, y, c).[14] Thus for

[14]To check this, note first that for $0 \leqq x < 1$, $g(x) = f(x) - x$ is strictly increasing in x, since $g'(x) = f'(x) - 1 > 0$ for $0 \leqq x < 1$. Also, for $0 \leqq x < 1$, $g(x) < f(1) - 1 = 1$. We observe, that (i) if $x'_t \leq 1$ for some $t > 3$, then $c'_{t+1} \geqq c_{t+1} > 1$ implies that $x'_{t+1} < 1$. (ii) If $x'_t < 1$ for some t, (say $x_t = 1 - \eta$, $\eta > 0$), then there is $\hat{\eta} > 0$, such that $x'_{s+1} \leqq x'_s - \hat{\eta}$ for all $s \geqq t$. To check this, let $(f(x'_t) - x'_t = 1 - \hat{\eta}$. We know that $\hat{\eta} > 0$. Then

$$x'_{t+1} = f(x'_t) - c'_{t+1} = f(x'_t) - x'_t - c'_{t+1} + x'_t \leqq (1 - \hat{\eta}) - 1 + x'_t = x'_t - \hat{\eta}.$$

Similarly,

$$x'_{t+2} = f(x'_{t+1}) - c'_{t+2} = f(x'_{t+1}) - x'_{t+1} - c'_{t+2} + x'_{t+1}$$
$$\leqq f(x'_t) - x'_t - c'_{t+2} + x'_{t+1} \leqq (1 - \hat{\eta}) - 1 + x'_{t+1} = x'_{t+1} - \hat{\eta}.$$

Exactly, the same argument can be repeated for each successive period. (i) and (ii) imply that if $x'_t \leq 1$ for some $t > 3$, then $x'_T < 0$ for sufficiently large T, a contradiction to the feasibility of (x', y', c').

$$t \geqq t_2 = \max(t_1, 3), (x_t - \varepsilon_t) > 1,$$

i.e., for

$$t \geqq t_2, \varepsilon_t < x_t - 1 = \frac{(2t+1)^2}{(t+1)^4}.$$

Hence,

$$p_t \varepsilon_t \leqq (t^2) \frac{(2t+1)^2}{(t+1)^4} \leqq 4 \quad \text{for} \quad t \geqq t_2.$$

Also, using (6) and (A.3),

$$p_{t+1}\varepsilon_{t+1} \geqq p_t \varepsilon_t \quad \text{for} \quad t \geqq t_1,$$

so

$$\inf_{t \geqq t_1} p_t \varepsilon_t = \delta > 0.$$

Using Taylor's expansion on (6) for

$$t \geqq t_2, \varepsilon_{t+1} = f(x_t) - f(x_t - \varepsilon_t) = f'(x_t)\varepsilon_t \left[1 + \frac{1}{2} \frac{[-f''(\S_t)]}{f'(x_t)} \varepsilon_t \right]$$

where

$$x_t' \leqq \S_t < x_t.$$

That is,

$$p_{t+1}\varepsilon_{t+1} = p_t \varepsilon_t \left[1 + \frac{1}{2} \frac{[-f''(\S_t)]}{p_t f'(x_t)} p_t \varepsilon_t \right].$$

Now, since

$$1 < x_t - \varepsilon_t \leqq \S_t \leqq x_t,$$

so

$$0 < \S_t - 1 \leqq x_t - 1 \quad \text{and} \quad \{1/[2(\S_t - 1)^{1/2}]\} \geqq (t+1)^2/[2(2t+1)].$$

Hence,

$$[-f''(\S_t)] = \{1/[2(\S_t - 1)^{1/2}]\} \geqq (t+1)^2/[2(2t+1)].$$

Thus,

$$p_{t+1}\varepsilon_{t+1} \geqq p_t \varepsilon_t \left[1 + \frac{\delta(t+1)^2}{4t^2(2t+1)} \right] \geqq p_t \varepsilon_t \left[1 + \frac{\delta}{4(2t+1)} \right].$$

Since $\sum_{t=t_2}^{\infty} 1/(2t+1)=\infty$, so $p_t\varepsilon_t \to \infty$ as $t\to\infty$. But this contradicts the fact that $p_t\varepsilon_t \leqq 4$ for $t \geqq t_2$. Hence (x, y, c) is efficient.

The problem, in this framework, is that the right-hand inequality of Condition S* fails. To check this, suppose the right-hand inequality is met for some λ, M satisfying $0<\lambda\leqq 1$, $0<M<\infty$. Then, choosing $\varepsilon_t=\lambda x_t/8t^2$ for $t\geqq 1$, we have, for $t \geqq 1$.

$$\frac{f(x_t)-f(x_t-\varepsilon_t)}{\varepsilon_t f'(x_t)}-1=\frac{1}{2}\frac{[-f''(\zeta_t)]}{f'(x_t)}\varepsilon_t=\frac{1}{2}\frac{[-f''(\zeta_t)]x_t}{f'(x_t)}\frac{\varepsilon_t}{x_t}\frac{W(x_t)}{W(x_t)}$$

$$=\frac{1}{2}\frac{[-f''(\zeta_t)]x_t}{f'(x_t)W(x_t)}\frac{\varepsilon_t}{x_t}W(x_t)=M_t\left\{\frac{\varepsilon_t}{x_t}W(x_t)\right\}.$$

Since $0<\varepsilon_t<\lambda x_t$, so

$$M_t \leqq M.$$

However, since

$$x_t \geqq \zeta_t \geqq x_t-\varepsilon_t=x_t\left(1-\frac{\lambda}{8t^2}\right)\geqq x_t\left(1-\frac{1}{8t^2}\right)$$

$$=\left[1+\frac{(2t+1)^2}{(t+1)^4}\right]\left[1-\frac{1}{8t^2}\right]\geqq\left[1+\frac{1}{4t^2}\right]\left[1-\frac{1}{8t^2}\right]>1,$$

so

$$0<(\zeta_t-1)\leqq(x_t-1),$$

and

$$[-f''(\zeta_t)]=\frac{1}{2(\zeta_t-1)^{1/2}}\geqq\frac{(t+1)^2}{2(x_t-1)^{1/2}}=\frac{(t+1)^2}{2(2t+1)}.$$

Since $x_t \geqq 1, f'(x_t)\leqq 1$, and $W(x_t)\leqq 1$, so

$$M_t \geqq \frac{(t+1)^2}{4(2t+1)}.$$

Thus, $M_t \to \infty$ as $t\to\infty$, and $M_t\leqq M<\infty$ is contradicted. Thus, the right-hand inequality of Condition S* is violated, in this framework, for the feasible program (x, y, c).

6. Remarks on characterizing inefficiency in a weakly productive case

I shall discuss here an example of a technology for which none of the criteria, proposed so far in the literature, suffice to identify inefficiency. However, Theorem 2 does provide a complete characterization of inefficiency for this case.

Consider $f(x)=x^{\alpha}+x$, for $x\geqq 0 (0<\alpha<1)$. If (1), (2) are considered to be derived from the standard Neo-classical growth equation (x, y, c are interpreted as per worker capital, gross-output, and consumption, respectively, and f a gross-output function), then this parametric form arises if the net-output function (defined on capital and labor inputs) is Cobb–Douglas, labor force is stationary and capital is fully durable.

Clearly, $f(x)$ satisfies (A.1)–(A.4). Also, recall from Remark 1 following Theorem 2, that feasible programs generated by f satisfy Condition S*. Hence inefficiency of any feasible program can be completely characterized by Theorem 2, i.e., by (8) and (23).

However, none of the three proposed criteria in the literature can identify inefficiency as the following examples demonstrate.

(i) Consider the feasible program $(\bar{x}, \bar{y}, \bar{c})$ given by $\bar{x}_0=1$, $\bar{x}_{t+1}=f\bar{x}_t)$ for $t\geqq 0$. This program is clearly inefficient. However, since $f'(\bar{x}_t)\geq 1$ for $t\geq 0$, $\sum_{t=0}^{T}(1/\bar{p}_t)\rightarrow\infty$, as $T\rightarrow\infty$, and (25) fails.

(ii) Consider the same program as in (i). Then, for $t\geq 0$, we have

$$(\bar{x}_{t+2}-\bar{x}_{t+1})=f(\bar{x}_{t+1})-f(\bar{x}_t)\leq f'(\bar{x}_t)(\bar{x}_{t+1}-\bar{x}_t).$$

By iteration,

$$(\bar{x}_{t+2}-\bar{x}_{t+1})\leqq \prod_{s=0}^{t} f'(\bar{x}_s)(\bar{x}_1-\bar{x}_0)=(\bar{x}_1-\bar{x}_0)/\bar{p}_{t+1}.$$

Hence,

$$\bar{x}_{t+2}\leq\bar{x}_{t+1}+(\bar{x}_1-\bar{x}_0)/\bar{p}_{t+1},$$

and by iteration on this step,

$$\bar{x}_{t+2}\leq\bar{x}_1+\sum_{s=0}^{t}[(\bar{x}_1-\bar{x}_0)/\bar{p}_{s+1}];$$

i.e.,

$$\bar{p}_{t+2}\bar{x}_{t+2}\leq\bar{p}_{t+2}\bar{x}_1+(\bar{x}_1-\bar{x}_0)(t+1)\leq t+3.$$

Hence,

$$\sum_{t=0}^{T} (1/\bar{p}_t\bar{x}_t) \to \infty, \text{ as } T \to \infty,$$

and (26) fails.

(iii) Consider the feasible program (x, y, c) given by $x_0 = 1$, and $x_{t+1} = f'(x_t)x_t$ for $t \geq 0$. Then $p_t x_t = 1$ for $t \geq 0$. And

$$W(x_t) = (1-\alpha)/(1+x_t^{1-\alpha}) \geq \tfrac{1}{2}(1-\alpha)/x_t^{1-\alpha}.$$

Now,

$$\sum_{t=0}^{T} (1/x_t^{1-\alpha}) \to \infty \text{ as } T \to \infty.$$

Otherwise, if

$$\sum_{t=0}^{T} (1/x_t^{1-\alpha}) < \infty,$$

then using the fact that

$$x_{t+1} = f'(x_t)x_t = (\alpha x_t^{\alpha-1} + 1)x_t = x_t[1 + (\alpha/x_t^{1-\alpha})],$$

we have,

$$x_{t+1} = \prod_{s=0}^{t} x_0[1 + (\alpha/x_t^{1-\alpha})] < \infty.$$

Then, there is $\beta > 0$, such that $f'(x_t) \geqq (1+\beta)$ for $t \geq 0$, and $p_t x_t \to 0$ as $t \to \infty$, a contradiction. Hence, $\sum_{t=0}^{T} (1/x_t^{1-\alpha}) \to \infty$ as $T \to \infty$, and so $\sum_{t=0}^{T} W(x_t) \to \infty$, as $T \to \infty$. Thus, (23) is violated (since $p_t x_t = 1$ for $t \geq 0$), and (x, y, c) is efficient. However, (8) is satisfied. Hence (8) fails to identify inefficiency.

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